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Optimal Selection of Weighting Matrices in Integrated Design of Structures/Controls

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A methodology is developed for the optimal selection of state and input weighting matrices, Q and R, respectively, of the linear quadratic regulator (LQR) method in the integrated design of structures/controls. An optimal control problem is set up in such a way that design variables are the diagonal entries of Q and R; the objective function is the trace of the solution matrix to the algebraic Riccati equation of the LQR method, P matrix; and constraints are imposed on the closed-loop eigenvalues to satisfy minimum stability conditions for the control system. The procedure finds the optimal diagonal Q and R that enables the actively controlled system to meet the prespecified stability and performance bounds. Furthermore, the resulting Q and R yield the minimum possible performance index and hence the control effort is substantially reduced. The proposed method is integrated with a substructure decomposition scheme which results in substantial savings on the numerical computations with very little loss in the accuracy of the original system response. It is found that for trusslike space structures, the proposed optimization scheme is mostly affected by the changes in the diagonal terms of R and the changes in the velocity diagonal terms of Q of the controlled system. The method is expected to be very useful for large-scale systems and is illustrated with the help of two example problems.

Nomenclature

- A = system matrix
- A_{cl} = closed-loop system matrix
- $\{A\}_k$ = set of internal degrees of freedom for the kth substructure
- B = input matrix
- $\{B\}_k$ = set of boundary degrees of freedom for the kth substructure
- C = output matrix
- C_d = damping matrix
- f_k = actuator force generated by the kth substructure
- G = gain matrix
- g_i = ith constraint function
- $\{I\}_k$ = set of original internal degrees of freedom for the kth substructure
- J = quadratic performance index
- K = stiffness matrix
- M = mass matrix
- P = solution matrix to
- Q = state weighting matrix
- R = input weighting matrix
- r = number of substructures
- s = number of substructures neighboring the kth substructure
- T_k = transformation matrix for the kth substructure
- tr(P) = trace of P
- u = input vector
- u_i = ith right eigenvector
- v_j = jth left eigenvector
- x = displacement vector
- y = output vector
- z = state vector
- z_0 = initial state vector
- λ_i = ith closed-loop eigenvalue
- λ_{iu} = ith closed-loop eigenvalue upper bound
- μ_i = ith performance bound corresponding to inputs
- σ_i = ith performance bound corresponding to states

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Introduction

WITH the ever increasing requirements of stability and performance characteristics imposed on systems, the subject of actively controlled systems in general and structures in particular has drawn much attention in recent years.[1,2] Optimization of actively controlled structures using different multiobjective optimization techniques was conducted by Rao.[3,4] When sufficient control energy is present, the closed-loop system poles can be virtually placed anywhere in the complex left-half plane by the active control method, hence it is especially attractive for the vibration suppression of lightweight flexible structures. In some cases, it may be advantageous to combine the active control with the passive control to improve the stability margin of the controlled system and also reduce the control cost. There has been a steady increase in research activities related to the integrated active and passive control design in the past few years.[5–7]

The linear quadratic regulator (LQR) method is a structured control method that is often employed in the active control of deterministic systems.[8,9] The LQR method minimizes a quadratic performance index with the constraints being the equations of motion (EOM) of the system in state-space form. The quadratic performance index provides a measure of total system strain, kinetic and potential energies, and its minimum value is dependent on the choices of the weighting matrices Q and R. The LQR method requires only Q and R be semipositive definite and positive definite, respectively, and hence they can be chosen from a wide range of possible candidates. Since the choices of Q and R affect the control cost, it is crucial to select these matrices to yield the minimum control cost while the prespecified performance and stability criteria are satisfied.

Because of the importance of the proper selection of Q and R, several works have appeared in the literature dealing with the subject, but most methods are concerned with the pole assignment.[10,11] The relationships between the weights and sensitivity were used by Athans[12] and by McEwen and Looze[13] to determine the proper weights. The weighting matrix selection based on satisfying input amplitude and variance constraints in a linear quadratic Gaussian (LQG) problem was suggested by Makila et al.[14] and Skelton and DeLorenzo.[15] The selection of weighting matrices in the definition of quadratic performance index and its implication in control of structures was discussed by Venkayya and Tischler.[16] The choice of the weighting matrices in the deterministic linear quadratic impulse (LQI) problems for continuous and discrete systems was presented by Zhu and Skelton.[17] Another weight selection algorithm of LQR problems for robust pole assignment was developed by Liebst and Robinson.[18] The algorithm chooses Q and R to place the closed-loop poles at a set of desired pole locations. The selection of the weighting matrices with some diagonal weights that achieve a specified pole location was addressed by Ohta et al.[19]

Since it is neither practical nor feasible to consider the full dynamic model of large systems, the decentralized control is especially important in the active control of large flexible structures. The concept of substructural controller design permits the design of controllers at component level and makes the global controller design of large flexible structures computationally attractive. A substructural control approach, called the controlled component synthesis, was developed by Young[9] in which an interlocking control concept was used to minimize the motion of the nodes adjacent to the boundaries of substructures. In another substructural control technique which was presented by Pan et al.,[8] the LQR method was applied at the substructure level in an attempt to balance the interaction forces between the substructures.

In this work, it is shown that the minimization of the quadratic performance index for some choices of Q and R is proportional to the trace of P matrix, the solution matrix to the algebraic Riccati equation (ARE). Hence, if the trace of P, tr(P), with the diagonal entries of Q and R taken as design variables, is minimized, the resulting Q and R yield the minimum of all the minimum quadratic performance indices. The objectives of this work are then stated as follows: 1) to find the minimum of all the minimum quadratic performance indices and hence the minimum possible control cost for a system; 2) to meet all the prespecified performance and stability bounds required of the system; and 3) to integrate the procedure with a substructural control technique to substantially save the computational time.

A control optimization problem is formulated, the global controller of the system is designed using the complete and substructural models of the system, and the computational advantage of the substructural model over the complete model is noted.

Formulation of the Problem

Consider the following multiobjective control optimization problem:

Minimize

$$J = [\int_0^\infty (y^TQy + u^TRu)\,\mathrm{d}t]$$

subject to

$$\dot{z} = Az + Bu, \qquad y = Cz, \qquad u = -Gz$$

$$Q_{ij\,\min} = \frac{1}{\sigma_{i\,\max}^2}\delta_{ij}, \qquad i, j = 1, \ldots, 2n$$

$$R_{ij\,\min} = \frac{1}{\mu_{i\,\max}^2}\delta_{ij}, \qquad i, j = 1, \ldots, m$$

$$g_i \leq 0, \qquad i = 1, \ldots, n_g \tag{1}$$

where δ_{ij} is the Kronecker delta, $2n$ and m are the degrees of freedom (DOF) of the system in state space and number of actuators in the system, respectively, and n_g is the number of constraints. If $\sigma_{i\,\max}^2$ and $\mu_{i\,\max}^2$ denote the ith prespecified performance bounds, they may be defined as

$$\int_0^\infty y_i^2\,\mathrm{d}t \leq \sigma_{i\,\max}^2 \qquad \int_0^\infty u_i^2\,\mathrm{d}t \leq \mu_{i\,\max}^2 \tag{2}$$

The physical meaning of the problem set by Eqs. (1) and (2) is that the root-mean-squared (rms) values of output states and inputs are constrained. The rms constraints are usually of interest in engineering problems. Such constraints may reflect the safety limits of a structure, since deflections exceeding a certain value will exceed the stress limitations of the structure or exceed the linear elastic region causing the dynamic structure to behave in a nonlinear manner. Thus, the suggested multiobjective control scheme has much practical significance. The solution to the problem is given by

$$u = -Gz, \qquad G = R^{-1}B^TP \tag{3}$$

where P is found from

$$PA + A^TP - PBR^{-1}B^TP + C^TQC = 0 \tag{4}$$

which is known as the ARE. The average value of J and J_{av} in the optimization problem can be proved to be proportional to tr(P) as follows: It is well known[5] that

$$J = z_0^TPz_0 = \sum_{i=1}^{2n} P_{ii}z_i^2 + 2\sum_{\substack{i,j=1\\ i\neq j}}^{2n} P_{ij}z_iz_j \tag{5}$$

where z_0, z at $t = 0$, is assumed as $z_0^T = [z_1\ z_2\ \cdots\ z_{2n}]$. J_{av} is found as

$$J_{av} = \left\{\int_S \left[\sum_{i=1}^{2n} P_{ii}z_i^2 + 2\sum_{\substack{i,j=1\\ i\neq j}}^{2n} P_{ij}z_iz_j\right]\mathrm{d}A\right\}\bigg/\int_S \mathrm{d}A \tag{6}$$

where S denotes the surface enclosed by the vector z_0. In Eq. (6)

$$\int_S \sum_{i=1}^{2n} P_{ii}z_i^2\,\mathrm{d}A = \sum_{i=1}^{2n} P_{ii}\int_S z_i^2\,\mathrm{d}A = \sum_{i=1}^{2n} P_{ii}V \tag{7}$$

where V is the volume formed by z_0. In Eqs. (6) and (7), if V is the volume of a unit ball, for example, then S is the surface defined by $\|z_0\| = 1$. Furthermore,

$$\int_S \sum_{\substack{i,j=1\\ i\neq j}}^{2n} P_{ij}z_iz_j\,\mathrm{d}A = \sum_{\substack{i,j=1\\ i\neq j}}^{2n} P_{ij}\int_S z_iz_j\,\mathrm{d}A = 0 \tag{8}$$

due to the divergence theorem. Lastly,

$$\int_S \mathrm{d}A = 2nV \tag{9}$$

Hence Eq. (6) becomes

$$J_{av} = \left(\sum_{i=1}^{2n} P_{ii}V\bigg/2nV\right) = \frac{1}{2n}\sum_{i=1}^{2n} P_{ii} = \frac{1}{2n}\mathrm{tr}(P) \tag{10}$$

Thus the problem of Eq. (1) can be posed as follows: Minimize tr(P), subject to

$$PA + A^TP - PBR^{-1}B^TP + C^TQC = 0$$

$$Q_{ij\,\min} = \frac{1}{\sigma_{i\,\max}^2}\delta_{ij}, \qquad i, j = 1, \ldots, 2n$$

$$R_{ij\,\min} = \frac{1}{\mu_{i\,\mathrm{ma}}^2}\delta_{ij}, \qquad i, j = 1, \ldots, m$$

$$g_i \leq 0, \qquad i = 1, \ldots, n_g \tag{11}$$

where the ith constraint function g_i places an upper bound on the ith eigenvalue of the closed-loop system (i.e., the eigenvalue of $A_{cl} = A - BG$). Hence g_i becomes

$$g_i = \lambda_i - \lambda_{iu} \leq 0, \qquad i = 1, \ldots, 2n \tag{12}$$

It can be shown that λ_i depends on the choices of Q and R, and hence the preceding optimization problem is well defined. The characteristic polynomial of A_{cl}, ΔA_{cl}, is found as[20]

$$\Delta A_{cl}(s) = (-1)^{2n}\Delta A(s)\Delta A(-s)\,|I + QL(s)R^{-1}L^T(-s)| \tag{13}$$

where s is the Laplace transformation variable and $L(s) = C(sI - A)^{-1}B$ and $\Delta A(s) = |sI - A|$. Note that the characteristic polynomial of A_{cl} is dictated by the choices of Q and R. Equations (11) set a procedure that yields the minimum possible quadratic performance index and meets the specified performance and stability bounds by minimizing J with respect to the diagonal terms of Q and R.

Design Sensitivity Analysis

Gradient calculations of the objective and constraint functions with respect to the diagonal terms of Q and R are performed as follows.

The eigenvalue problem of A_{cl} is defined as

$$A_{cl}u_i = \lambda_i u_i, \qquad A_{cl}^T v_j = \lambda_j v_j, \qquad i, j = 1, \ldots, 2n \quad (14)$$

where $\lambda_i(\lambda_j)$ are the eigenvalues (assumed as distinct), and u_i and v_j are the right and left eigenvectors of A_{cl}. After normalization

$$u_i^T v_j = v_j^T u_i = \delta_{ij} \quad (15)$$

After some manipulation

$$\frac{\partial \lambda_j}{\partial Q_{ii}} = \frac{\partial g_j}{\partial Q_{ii}} = v_j^T \frac{\partial A_{cl}}{\partial Q_{ii}} u_j, \qquad \frac{\partial \lambda_j}{\partial R_{ii}} = \frac{\partial g_j}{\partial R_{ii}} = v_j^T \frac{\partial A_{cl}}{\partial R_{ii}} u_j \quad (16)$$

where

$$\frac{\partial A_{cl}}{\partial Q_{ii}} = -BR^{-1}B^T \frac{\partial P}{\partial Q_{ii}}$$

$$\frac{\partial A_{cl}}{\partial R_{ii}} = \frac{1}{R_{ii}^2} B_i B_i^T P - BR^{-1}B^T \frac{\partial P}{\partial R_{ii}} \quad (17)$$

and B_i is the ith column of the B matrix. Furthermore,

$$\frac{\partial[\mathrm{tr}(P)]}{\partial Q_{ii}} = \mathrm{tr}\left(\frac{\partial P}{\partial Q_{ii}}\right), \qquad \frac{\partial[\mathrm{tr}(P)]}{\partial R_{ii}} = \mathrm{tr}\left(\frac{\partial P}{\partial R_{ii}}\right) \quad (18)$$

where

$$A_{cl}^T \frac{\partial P}{\partial Q_{ii}} + \frac{\partial P}{\partial Q_{ii}} A_{cl} + C_i^T C_i = 0 \quad (19)$$

and

$$A_{cl}^T \frac{\partial P}{\partial R_{ii}} + \frac{\partial P}{\partial R_{ii}} A_{cl} + P \frac{B_i B_i^T}{R_{ii}^2} P = 0 \quad (20)$$

which are Lyapunov equations for $(\partial P/\partial Q_{ii})$ and $(\partial P/\partial R_{ii})$, respectively. In Eq. (19), C_i denotes the ith row of the output matrix C.

Substructural Controller Design for Flexible Structures

It is assumed that the flexible structure is decomposed into r substructures, where the kth substructure is neighbored by s substructures. Note that in the subsequent discussions, the subscript k refers to the kth substructure. Let $\{I\}_k$ denote the set of original internal DOF of the kth substructure, $\{B_1\}_k$ represent the set of boundary DOF between the kth and $(k+1)$th substructures, and so on. The EOM of the kth substructure can be represented as

$$M_k \ddot{x}_k + C_{dk} \dot{x}_k + K_k x_k = D_k u_k \quad (21)$$

where x_k and u_k denote the vectors of displacement and control input, respectively; and M_k, C_{dk}, K_k, and D_k denote the mass, damping, stiffness, and input weighting matrices in the configuration space, respectively. The matrices M_k, C_{dk}, K_k, and D_k have to be summed properly to preserve the displacement compatibility of the whole structure.

The EOM of the kth substructure, Eq. (21), can be rearranged by grouping together the internal and boundary DOF of the substructure. The internal and boundary DOF of a substructure will vary depending on the surrounding substructure considered. Considering the kth substructure and its neighboring $(k+1)$th substructure, the set of internal DOF is given by $\{A_k\} = \{\{I_k\}, \{B_2\}, \ldots, \{B_s\}\}_k$ and the set of boundary DOF by $\{B_k\} = \{B_1\}_k$. Hence, the partitioned EOM for the kth substructure can be stated as

$$\begin{bmatrix} M_{AA} & M_{AB} \\ M_{BA} & M_{BB} \end{bmatrix}_k \begin{Bmatrix} \ddot{x}_A \\ \ddot{x}_B \end{Bmatrix}_k + \begin{bmatrix} C_{dAA} & C_{dAB} \\ C_{dBA} & C_{dBB} \end{bmatrix}_k \begin{Bmatrix} \dot{x}_A \\ \dot{x}_B \end{Bmatrix}_k + \begin{bmatrix} K_{AA} & K_{AB} \\ K_{BA} & K_{BB} \end{bmatrix}_k \begin{Bmatrix} x_A \\ x_B \end{Bmatrix}_k = \begin{bmatrix} D_{AA} & D_{AB} \\ D_{BA} & D_{BB} \end{bmatrix}_k \begin{Bmatrix} u_A \\ u_B \end{Bmatrix}_k \quad (22)$$

By using the Guyan static condensation method, the following relation can be written:

$$\begin{Bmatrix} x_A \\ x_B \end{Bmatrix}_k = \begin{bmatrix} -K_{AA}^{-1}K_{AB} \\ I_B \end{bmatrix}_k x_{Bk} = T_k x_{Bk} \quad (23)$$

where T_k is the transformation matrix for the kth substructure while considering the interaction with the $(k+1)$th substructure. By defining proper sets for $\{A_k\}$ and $\{B_k\}$, the transformation matrices between the kth substructure and the other substructures can be determined using Eq. (23).

The EOM of the kth substructure in state-space form are given by

$$\dot{z}_k = A_k z_k + F'_{Bk} z_k + B_k u_k = A'_k z_k + B_k u_k \quad (24)$$

where A_k and B_k are the system and state-space input matrices, respectively; u_k is the vector of control input generated within the kth substructure, and z_k is the state vector for the kth substructure defined as

$$z_k = \{x_k \ \dot{x}_k\}^T = \{x_{Ak} x_{Bk} \dot{x}_{Ak} \dot{x}_{Bk}\}^T \quad (25)$$

and also

$$F'_{Bk} z_k = \sum_{i=1}^{s} \begin{Bmatrix} 0 \\ M_k^{-1} F_{Bik} \end{Bmatrix} \quad (26)$$

where F_{Bik} is the actuator (controller) force generated by the ith substructure surrounding the kth substructure. Note that F_{Bik} denotes the actuator force passed from the previous iteration. The input vector u_k in Eq. (24) is found using the LQR method as

$$u_k = -G_k z_k = -[R^{-1}B^T P]_k z_k \quad (27)$$

where P_k satisfies the following ARE

$$P_k A'_k + A'^T_k P_k - P_k B_k R_k^{-1} B_k^T P_k + C_k^T Q_k C_k = 0 \quad (28)$$

Recall that the input weighting matrix of the kth substructure is defined as

$$B_k = \begin{bmatrix} 0 \\ M_k^{-1} D_k \end{bmatrix} \quad (29)$$

where D_k is the input weighting matrix of the kth substructure in the configuration space. Postmultiplying Eq. (29) by Eq. (27) yields

$$B_k u_k = \begin{bmatrix} 0 \\ -M_k^{-1} D_k G_k \end{bmatrix} z_k \quad (30)$$

In the current iteration, the actuator force which is generated within the kth substructure to balance the interacting forces from the surrounding substructures (and to suppress vibration) can be determined using Eq. (30) as

$$f_k = \begin{Bmatrix} f_{Ak} \\ f_{Bk} \end{Bmatrix} = [-M_k M_k^{-1} D_k G_k] z_k = [-D_k G_k] \begin{Bmatrix} x_{Ak} \\ x_{Bk} \\ \dot{x}_{Ak} \\ \dot{x}_{Bk} \end{Bmatrix} \quad (31)$$

where f_{Ak} and f_{Bk} are the actuator forces at the internal and boundary DOF of the kth substructure, respectively. By defining proper sets of $\{A\}_k$ and $\{B\}_k$ for the kth substructure and each of its surrounding substructures, Eq. (23) can be used to condense the actuator force, f_k in Eq. (31), into the boundary DOF of the surrounding substructures. Consequently, Eq. (31) can be expressed as

$$f_k = \begin{Bmatrix} f_{Ak} \\ f_{Bk} \end{Bmatrix} = [-D_k G_k T'_k] \begin{Bmatrix} x_{Bk} \\ \dot{x}_{Bk} \end{Bmatrix} \tag{32}$$

where

$$T'_k = \begin{bmatrix} T_k & 0 \\ 0 & T_k \end{bmatrix} \tag{33}$$

Note that f_{Bki} which is the actuator force generated by the kth substructure and condensed into the boundary DOF of the ith surrounding substructures, is not same as $F_{Bki}(-F_{Bik})$ since they are obtained at two different iterations. At the final iteration, however, they are the same.

The following multiobjective optimization control problem is now posed using the substructural control method just developed:

Minimize

$$\sum_{k=1}^{r} [\mathrm{tr}(P_k)]$$

subject to

$$P_k A_k + A_k^T P_k - P_k B_k R_k^{-1} B_k^T P_k + C_k^T Q_k C_k = 0 \qquad k = 1, \ldots, r$$

$$Q_{ij\,\min} = \frac{1}{\sigma_{i\,\max}^2} \delta_{ij}, \qquad i, j = 1, \ldots, 2n_k, \quad k = 1, \ldots, r$$

$$R_{ij\,\min} = \frac{1}{\mu_{i\,\max}^2} \delta_{ij}, \qquad i, j = 1, \ldots, m_k, \quad k = 1, \ldots, r$$

$$g_i \leq 0, \qquad i = 1, \ldots, n_g \tag{34}$$

where $2n_k$ and m_k denote the DOF in state space and number of actuators for the kth substructure, respectively. The numerical procedure to solve Eqs. (34) is given as follows.

I) Initially set Q_0 and R_0.

II) $i = -1$.

III) $i = i + 1$ and $Q = Q_i$, $R = R_i$.

1) $k = 0$
2) $f_i = 0$
3) $k = k + 1$
4) Solve the ARE for the kth substructure according to the substructural control scheme just discussed. Note that $(Q_k)_i$ and $(R_k)_i$ are in the subspace of Q_i and R_i and are subject to the constraints as imposed by Eqs. (34).
5) $(f_k)_i = \mathrm{tr}(P_k)_i$
6) $f_i = f_i + (f_k)_i$
7) If $k \leq r$ go to step 3. Otherwise continue to step IV.

IV) Assemble the global closed-loop system and construct the constraint functions $(g_j)_i$, $j = 1, \ldots, n_g$.

V) At the ith step of optimization, the objective and constraint functions are f_i and $(g_j)_i$, $j = 1, \ldots, n_g$, respectively. Go to step III until the convergence of the optimization method.[21]

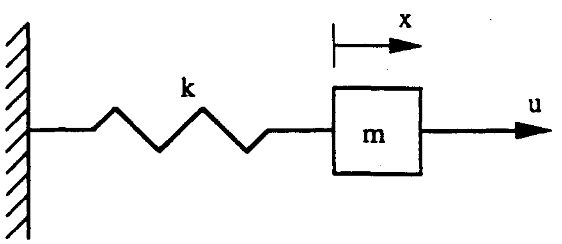

Fig. 1 Spring-mass system.

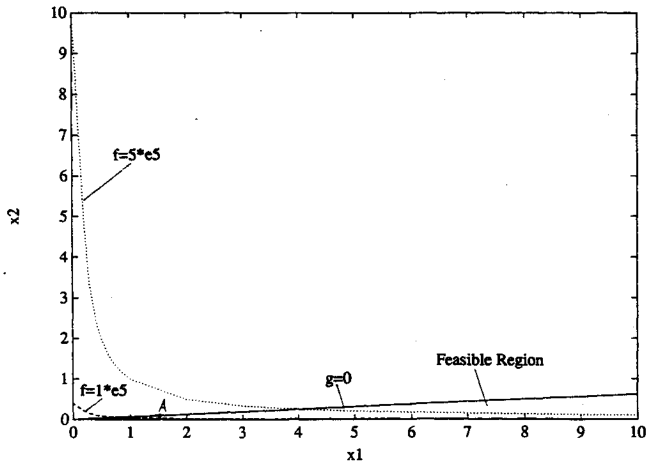

Fig. 2 Optimum point for spring-mass system.

In this work, the ARE is solved by the technique described by Kleinman.[22] The optimization method used in step V is the modified method of feasible directions developed by Vanderplaats.[21]

Numerical Examples

Mass-Spring System

The 2-DOF mass-spring system shown in Fig. 1 is first used as a simple example to clearly demonstrate the control optimization procedure developed earlier. Q is taken as a full matrix, and it is shown that only the diagonal terms of Q have dominant effect on the optimization procedure.

The EOM of the system is written as

$$m\ddot{x} + kx = u \tag{35}$$

After converting Eq. (35) into the standard state-space form, the system and input matrices A and B are found as

$$A = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} \tag{36}$$

Q and R are taken as

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix}, \qquad R = r \tag{37}$$

The control force, u in Eq. (35), is defined as

$$u = -g_1 x - g_2 \dot{x} \tag{38}$$

where the control gain matrix $G = [g_1\ g_2]$ is given by

$$G = \frac{1}{r} B^T P \tag{39}$$

After solving the ARE, the optimization procedure is set as follows:

Minimize

$$f = \mathrm{tr}(P) = k\sqrt{q_{22} r} - q_{12}$$

subject to

$$g = -\frac{1}{m}\sqrt{\frac{q_{22}}{r}} + 1 \leq 0$$

$$q_{22} \leq 0.1 \qquad r \leq 0.1 \tag{40}$$

where the constraint is imposed such that the real parts of the closed-loop eigenvalues of the system are smaller than or equal to -1. In obtaining Eqs. (40), it is assumed that $q_{11}/(k^2 r) < 1$. This is generally true, since the value of k for a spring (or a bar) will be very large. Note that, in Eqs. (40), the value of the first term in f, $k\sqrt{q_{22} r}$, will usually be much higher than that of q_{12}, and the constraint g is only the function of q_{22} and r. Hence, the optimization process will be heavily dependent on

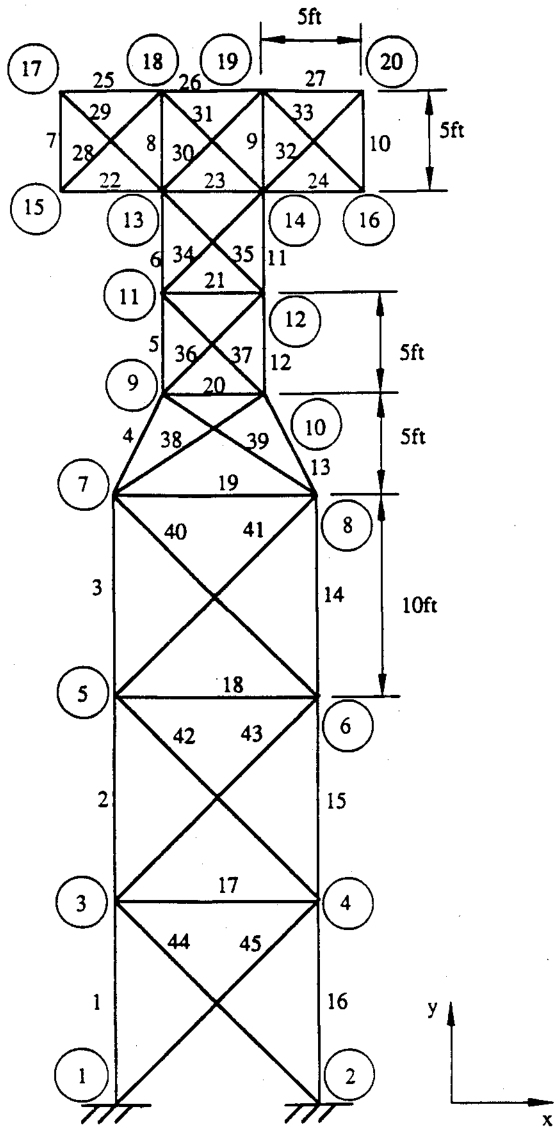

Fig. 3 45-bar truss.

q_{22} and r. This trend is also seen in the next example problem.

A bar with a cross-sectional area of $A = 1$ in.^{2} and a length of $L = 20$ in. is used to further illustrate the observations just made. The material properties are chosen as Young's modulus $E = 10^7$ psi and mass density $\rho = 0.1$ lb/in.^{3} Hence, the spring constant $k = AE/L = 500{,}000$ lb/in. and mass $m = \rho AL = 2$ lb. Using these values, the solution to the optimization problem given by Eqs. (40) is found and shown in Fig. 2. It can easily be seen from Fig. 2 that when the values of q_{22} and r decrease, the value of f decreases. Consequently, the optimal design point lies at the lower bound of q_{22}, which is marked as point A in Fig. 2.

45-Bar Truss

The 45-bar truss shown in Fig. 3 is used as a large-scale problem to further illustrate the control optimization procedure using complete and substructural models of the structure. The truss is made of aluminum with $E = 10^7$ psi and $\rho = 0.1$ lb/in.^{3}. It has 72 DOF in state space. The actuators are placed at nodes 7, 8, 9, 10, 13, 14, 18, and 19. The control forces can be generated in both x and y directions at these nodes. The Q and R, whose diagonal entries are taken as design variables, are diagonal matrices with sizes 72×72 and 16×16, respectively. Without loss of generality, the performance and stability requirements of the closed-loop system are stated such that all of the diagonal terms of Q and R are to be greater than or equal to 0.1, and the real parts of all of the closed-loop eigenvalues of the structure must be smaller than or equal to -1. Hence, for this problem, the number of design variables is 88 and the number of constraints is 72. The truss is decomposed into six substructures for the substructural control method. The cross-sectional areas of all members of the truss are taken as 2 in.^{2}.

As a first attempt, at the initial design, the Q and R matrices are set to $10^5 \times I_Q$ and I_R, where I_Q and I_R are the identity matrices with proper dimensions. The initial design (the initial choices of Q and R) has yielded the objective function [tr(P)] values of 1.45587×10^{11} and 9.05979×10^{10} for the complete model and substructural model, respectively. At the final iteration of the optimization, the objective function values have been noted as 5.50984×10^9 and 2.36725×10^9 for the complete and substructural models, respectively.

The displacement of the truss at node 20 in the y direction for both complete and substructural models at initial and final designs is shown in Figs. 4–7. The disturbance forces in the numerical simulations are taken as unit impulse and step forces. The actuator response at node 7 in the x direction resulting from the unit impulse force for both models at initial and final designs is also shown in Figs. 8 and 9. It is a straightforward task to show by Laplace or Fourier transformations that the unit step function has frequency content. Hence, due to the nature of Guyan reduction scheme, some

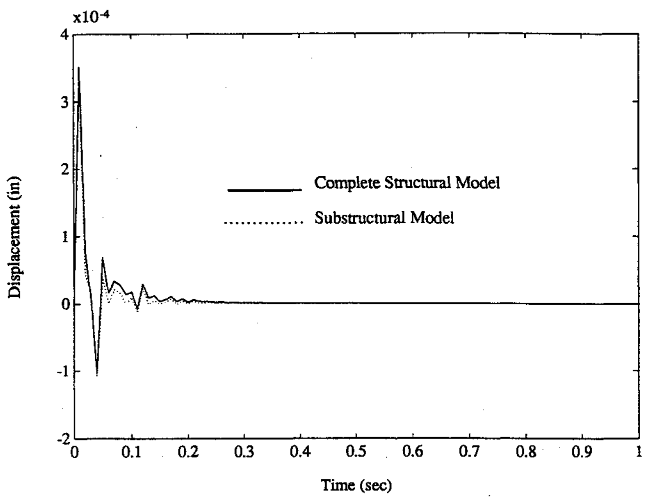

Fig. 4 Displacement of 45-bar truss at node 20 in *y* direction at initial design, response to unit impulse; complete structural model ——, substructural model ······.

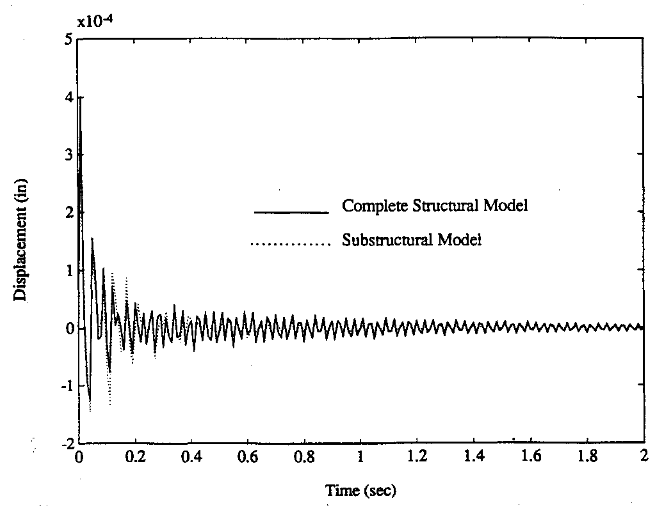

Fig. 5 Displacement of 45-bar truss at node 20 in *y* direction at final design, response to unit impulse; complete structural model ——, substructural model ······.

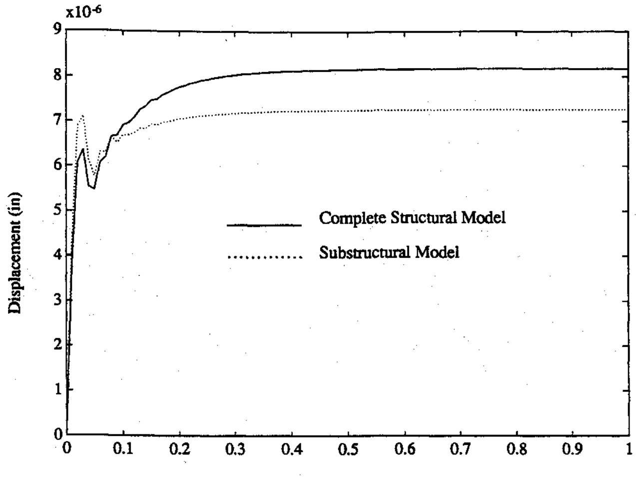

Fig. 6 Displacement of 45-bar truss at node 20 in y direction at initial design, response to unit step; complete structural model ——, substructural model ······.

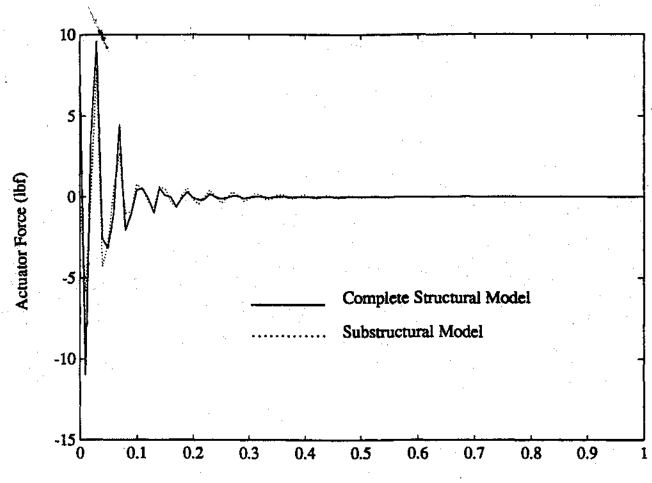

Fig. 8 Actuator force of 45-bar truss at node 7 in x direction at initial design, response to unit impulse; complete structural model ——, substructural model ······.

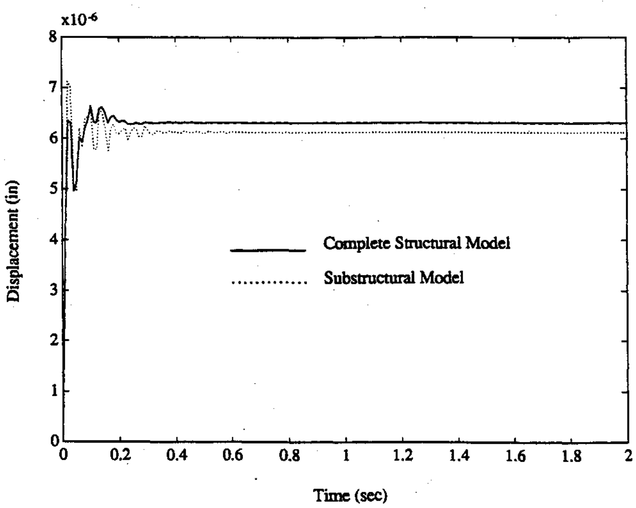

Fig. 7 Displacement of 45-bar truss at node 20 in y direction at final design, response to unit step; complete structural model ——, substructural model ······.

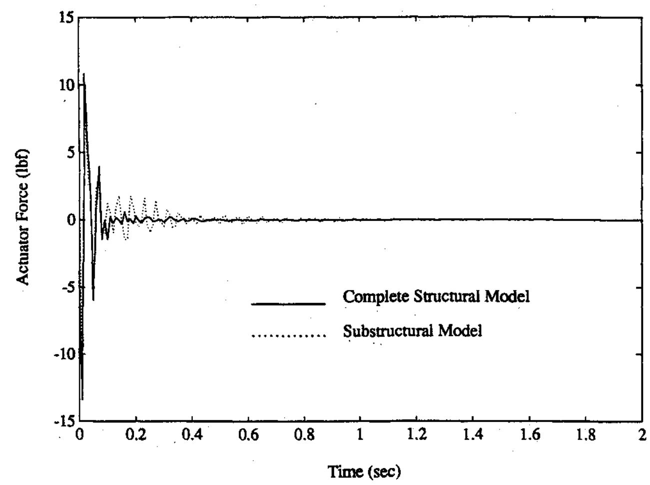

Fig. 9 Actuator force of 45-bar truss at node 7 in x direction at final design, response to unit impulse; complete structural model ——, substructural model ······.

discrepancy between the two models in the steady-state value of the structure response to the unit step force is expected. This can be seen in Figs. 6 and 7. However, the numerical experience with the substructural model and the simple observations made from Figs. 4–9 lead to the conclusion that the proposed multivariable control optimization scheme can be applied to most engineering problems.

As a second attempt, different initial values for Q and R are used to test the robustness of the scheme in reaching an optimum design from a different starting design. The robustness in this regard is expected to be affected by the optimization method used to solve the problems defined by Eqs. (11) and (34). At the initial design, the Q and R matrices are taken as $Q_{ii}=10^5$, $i=1, \ldots, 36$; $Q_{ii}=10^6$, $i=37, \ldots, 54$; $Q_{ii}=10^4$, $i=55, \ldots, 72$; $R_{ii}=10$, $i=1, \ldots, 8$; $R_{ii}=1$, $i=9, \ldots, 16$. For these initial choices of Q and R, the objective function values for the complete and substructural models have been noted as 4.19497×10^{11} and 3.76526×10^{11}, respectively. At the final designs obtained by the optimization algorithm, the objective function values have been recorded as 2.14690×10^{10} and 4.32325×10^9, respectively. Although the final designs obtained for both models with two different initial choices of Q and R slightly differ from each other, substantial savings in the objective function [tr(P)], and, hence, in the control effort have been achieved through the multivariable control optimization.

Three basic observations can be made from the optimization values and Figs. 4–9. First, it can be noted that a substantial saving in the value of the performance index and, hence, in the control effort is made through control optimization. Second, the use of the substructural control method greatly reduces the computational cost of the control optimization process without much loss on the accuracy of the system response. At the first attempt, the CPU times on a GOULD PN9080 machine for the control optimization process using a complete model and a substructural model have been noted as 35 h + 40 min + 41 s and 4 h + 30 min + 32 s, respectively. Hence the substructural control method resulted in almost 10 times less computational cost than the complete model. Third, for trusslike structures, the control optimization problem is most sensitive to the diagonal terms of Q corresponding to the velocity states of the truss and the diagonal terms of R.

Conclusion

An optimality-based strategy is presented to reduce the control cost of a system and to simultaneously meet the performance and stability bounds imposed on the system. A substructural control method in which the global controllers of large flexible structures are designed at the substructure level is also presented to reduce the computational cost involved with the control optimization process. It is shown that the minimization of the quadratic control effort is proportional to that of tr(P). Analytical expressions are derived to compute the first derivatives of tr(P) and those of the closed-loop eigenvalues with respect to the diagonal terms of the Q and R matrices, which are taken as design variables in the control optimization process.

It is clear from the foregoing discussion and illustrative examples that great savings in the control effort are possible through the approach proposed in this work. The substructural control method substantially reduces the computational cost of control optimization of large flexible structures. It is found that the quadratic performance index is most sensitive to the changes in the diagonal terms of R and those of Q associated with the velocity states. The approach is quite general and can be applied to any system with specified performance and stability bounds. The proposed method is expected to be particularly beneficial for large flexible structures.

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